

Euler's Formula: A Group-Theoretic Approach

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1 Motivation

It's well known that $e^{i\pi} + 1 = 0$, but why? Euler's formula is typically introduced to students in high-school as a part of a course in *Pre-Calculus*¹. However, one major issue arises: When students ask "why" they're told that it's outside the scope of the course. Indeed a proof of Euler's theorem is typically presented with the usage of power series in a standard Calculus II course. Proofs that don't require Calculus are either seldom known about much less taught. Thus, we present a proof through the lens of group theory.

Before proving Euler's theorem, we first construct the field of $\mathbb{C}$ and prove De'Moivre's theorem to motivate Euler's formula. We then define *groups* and *fields* to provide an introduction to abstract algebra that's needed for the proof. Then, we ultimately prove Euler's formula.

2 The Construction of $\mathbb{C}$

Let $i := \sqrt{-1}$ be the *imaginary*² unit. Its definition implies that $i^2 = -1$. We'll use this unit in our construction of the complex numbers. We define

$$\mathbb{C} := \{z \mid z := x + yi, \text{ for } x, y \in \mathbb{R}\}. \quad (1)$$

A complex number is defined by its two components; its real and imaginary parts. Using the construction in (1),

$$\operatorname{Re}(z) := x \text{ real part}$$

$$\operatorname{Im}(z) := y \text{ imaginary part.}$$

¹Some schools don't introduce the formula in courses meant to be taken before introductory calculus, but in competition math the formula is assumed to be known in divisions students would be placed in if they're taking Pre-Calculus or a course analogous to it.

² i is called an imaginary number since mathematicians of the time didn't see a geometric property of i that made it "real." Of course, we now know that i and complex numbers in general have many nice geometric properties.

Using the definitions above, we define the addition/subtraction of $z_1, z_2 \in \mathbb{C}$ as

$$z_1 \pm z_2 := (x_1 \pm x_2) + i(y_1 \pm y_2) \quad (2)$$

For a geometric understanding of addition/subtraction in $\mathbb{C}$, one may think of complex numbers as "vectors" in $\mathbb{R}^2$ using ideas built from vector addition/subtraction³. We may also define multiplication similar to how did for polynomials.

$$\begin{aligned} z_1 z_2 &:= x_1(x_2 + y_2 i) + y_1 i(x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + x_2 y_1 i + y_1 y_2 i^2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1). \end{aligned} \quad (3)$$

We define the *conjugate* of a complex number, $\bar{z}$, to be its reflection across the real-axis in the complex plane. Using geometry (or a rotation matrix if you're feeling extra quirky and unique), we find that

$$\bar{z} := x - yi. \quad (4)$$

Before defining division, notice that we can assign a length to a complex number using its length or *modulus*. Since the length of a complex number is positive and only 0 if $z = 0$, we can use the Pythagorean theorem to find that

$$|z| = \sqrt{x^2 + y^2} \quad (5)$$

where $|z|$ denotes the modulus. Notice that

$$z\bar{z} = |z|^2 \quad (6)$$

since

$$\begin{aligned} z\bar{z} &= (x + yi)(x - yi) \\ &= x^2 - xyi + xyi - y^2 i^2 \\ &= x^2 + y^2 \\ &= |z|. \end{aligned}$$

³This notion of treating complex numbers as vectors is deeper than presented in that one may define an *isomorphism* between $\mathbb{R}^2$ and $\mathbb{C}$ such that $\mathbb{R}^2 \cong \mathbb{C}$.

Using the notion of the modulus and conjugate of a complex number, we can turn division into multiplication as follows:

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} \quad (7)$$

$$= \frac{z_1 \bar{z}_2}{|z_2|^2}. \quad (8)$$

Notice that the denominator is always greater than or equal to 0, and so dividing two complex numbers turns into multiplication.

3 De'Moivre's Theorem

Before moving forward, we'll first prove De'Moivre's theorem;

$$(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta) \quad (9)$$

for $n \in \mathbb{Z}^+$. We'll prove it using induction. Notice that in the case of $n = 0$,

$$(\cos(\theta) + i \sin(\theta))^0 = \cos(0) + i \sin(0)$$

$$1 = 1$$

which is clearly true. We assume that (9) is true for $n = k$ where $k \in \mathbb{N}$. We then show that under this assumption, (9) is true for $n = k + 1$.

$$\begin{aligned} (\cos(\theta) + i \sin(\theta))^{k+1} &= (\cos(\theta) + i \sin(\theta)) (\cos(\theta) + i \sin(\theta))^k \\ &= (\cos(\theta) + i \sin(\theta)) (\cos(k\theta) + i \sin(k\theta)) \\ &= \cos(\theta) \cos(k\theta) + i \cos(\theta) \sin(k\theta) + i \cos(k\theta) \sin(\theta) + i^2 \sin(\theta) \sin(k\theta) \\ &= \underbrace{(\cos(\theta) \cos(k\theta) - \sin(\theta) \sin(k\theta))}_{\cos((k+1)\theta)} + i \underbrace{(\cos(\theta) \sin(k\theta) + \cos(k\theta) \sin(\theta))}_{\sin((k+1)\theta)} \\ &= \cos((k+1)\theta) + i \sin((k+1)\theta) \end{aligned}$$

and so the theorem is proved. Notice that parallels between the property that $x^a x^b = x^{a+b}$.

4 Groups

A group is a set G equipped with a binary operation we'll notate as $\cdot$. In short, if $g_1, g_2 \in G$, then $g_1 \cdot g_2 \in G$: This binary operation is just some mapping that takes two elements of the group, G , to produce another in G . A group is defined by the following:

1. $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$.
2. There exists an identity $e \in G$ such that $g \cdot e = e \cdot g = g$ for every $g \in G$.
3. There exists an inverse element, $g^{-1} \in G$, for every $g \in G$ such that $g \cdot g^{-1} = e$.