Selected Problems from Rudin's Book

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Problem 1

(1.5) Let A be a nonempty set of the real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup -A.$$

Solution

We begin by examining the infimum of A. Let $\alpha \in A$ be the number such that

$$\alpha \leq x, \quad \forall x \in A.$$

By definition, $\inf A = \alpha$. By construction of $-A$, the number $\beta \in -A$ such that

$$y \leq \beta, \quad \forall y \in -A,$$

is the supremum, $\sup -A = \beta$. Note that each element y corresponds to an element $x \in A$ such that $y = -x$. Without loss of generality, we rewrite the previous equation in terms of x and observe that the supremum of $-A$ is the number such that

$$-x \leq \beta, \quad \forall x \in A.$$

By properties of $\leq$, we have that

$$x \geq -\beta, \quad \forall x \in A.$$

By definition, $\inf A = -\beta$, and so, $\inf A = \alpha = -\beta = -\sup -A$.

Problem 2

(1.7) Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm* of y to the base b.)

(a) For any positive integer n, $b^n - 1 \geq n\,(b-1)$.

(b) Hence $b - 1 \geq n\left(b^{1/n} - 1\right)$.

(c) If $t > 1$ and $n > (b-1)\,/\,(t-1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n.

(e) If w is such that $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(g) Prove that this x is unique.

Solution

Problem 2 continued on next page. . .

Part (a)

Consider the factorization

$$b^n - 1 = (b-1)\left(b^{n-1} + \cdots + b + 1\right).$$

Because $b > 1$, $b^{n-1} + \cdots + b + 1 > n$, hence

$$b^n - 1 = (b-1)\left(b^{n-1} + \cdots + b + 1\right) \geq n\,(b-1).$$

Part (b)

Because $b > 1$ it follows that indeed $b^{1/n} > 1$. By Part (a),

$$\begin{aligned}\left(b^{1/n}\right)^n - 1 &\geq n\left(b^{1/n} - 1\right)\\ b - 1 &\geq n\left(b^{1/n} - 1\right).\end{aligned}$$

Part (c)

Rearranging the inequality, $n\,(t-1) > (b-1)$ implies

$$\begin{aligned}n\,(t-1) > (b-1) \geq n\left(b^{1/n} - 1\right) \implies n\,(t-1) &> n\left(b^{1/n} - 1\right)\\ t - 1 &> b^{1/n} - 1\\ t &> b^{1/n}.\end{aligned}$$

Part (d)

By assumption, it follows that $1 < yb^{-w}$. Choose some n such that

$$n > \frac{b-1}{yb^{-w} - 1}.$$

We can then apply Part (c) obtaining $b^{1/n} < yb^{-w}$ leading to $b^{w+(1/n)} < y$.

Part (e)

Similar to Part (d), it follows from the assumption that $b^w/y > 1$ and we can apply Part (c) in an analogous way in that we choose n such that

$$n > \frac{b-1}{\frac{b^w}{y} - 1}.$$

Thus,

$$b^{1/n} < b^w/y \implies yb^{1/n} < b^w \implies y < b^{w-(1/n)}.$$

Part (f)

Assume that $b^x > y$. By Part (e), there exists some n such that $b^x > b^{x-(1/n)} > y$. Because $b^x > b^{x-(1/n)}$ implies $x > x-(1/n)$, $x-(1/n)$ is an upper bound on A. However, this can't be because x is the least upper bound! Furthermore, assume $b^x < y$. By Part (d), then there exists some n such that $b^x < b^{x+(1/n)} < y$. It follows that $x < x+(1/n)$. Since $x+(1/n) \in A$, x can't be the least upper bound, however, x *is* the least upper bound by hypothesis, and so by the ordering of the real numbers $b^x = y$.

Part (g)

Because b^x is strictly increasing and the supremum of any ordered set is unique, it follows that if $b^x = y = b^w$, then $x = w$.

Problem 3

(1.8) Prove that no order can be defined in the complex field that turns it into an ordered field.

Solution

Lemma 1. *For any ordered field $\mathbb{F}$, for all $x \in \mathbb{F}$, $x^2 > 0$ if $x \neq 0$.*

Proof. Suppose $x > 0$. by the axioms of an ordered field, $x^x = xx > 0$. Suppose $x < 0$. Because $x^2 = xx = (-x)(-x)$ by the cancellation property of additive inverses. From our previous reasoning $x^2 > 0$. ☐

Suppose that there is such an order. We know that $1 = 1^2$, and so $0 < 1$ by Lemma 1. Adding -1 to both sides, $-1 < 0$. However, $i^2 = -1 > 0$ by Lemma 1, and so we have a contradiction! So, it must be the case that since $-1 < 0$ and $0 < -1$, $0 = -1$, but this can't be true by the field axioms. Thus, there exists no such order on $\mathbb{C}$ that makes $\mathbb{C}$ an ordered field.

Problem 4

(2.2) A complex number is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_n z^n + \cdots + a_1 z + a_0 = 0.$$

Prove that the set of all algebraic numbers is countable.

Solution

We'll assume the fundamental theorem of algebra to be true and note that for each set of coefficients (not all zero) there are n roots of the associated polynomial. Associate to such a polynomial an $n+1$-tuple $(a_0, \ldots, a_n)$. Because $P_{\mathbb{C}}^n(\mathbb{Z}) \cong \mathbb{Z}^{n+1}$ and $\mathbb{Z}^{n+1}$ is countable then so is $P_{\mathbb{C}}^n(\mathbb{Z})$. Taking the union of all such polynomials we have $\bigcup\limits_{n \in \mathbb{N}} P_{\mathbb{C}}^n(\mathbb{Z})$ which is of course a countable set. By the fundamental theorem of algebra, every n^{th} degree polynomial has exactly n roots. Thus, the collection of all roots in $\mathbb{C}$ of all polynomials over $\mathbb{Z}$ is a countable union of finite sets which must be countable.

Problem 5

(2.3) Prove that there exist real numbers which are not algebraic.

Solution

Let $\mathbb{A}$ denote the set of algebraic numbers. If all real numbers are algebraic then there exists some surjective map $f : \mathbb{A} \to \mathbb{R}$. However, since $\mathbb{A}$ is countable and $\mathbb{R}$, there exists no such f. So, there exists some $x \in \mathbb{R}$ that's not algebraic.

Problem 6

(2.12) Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, \ldots$. Prove that K is compact directly from the definition.

Solution

Let $\{V_\alpha\}$ be some open cover of K. Then there exists some V_{α^*} such that $0 \in V_{\alpha^*}$. Because each V_α is an open set, V_{α^*} is an open set, hence there exists some $\varepsilon > 0$ such that $B_\varepsilon(0) \subset V_{\alpha^*}$. By the Archimedian property of K as a subset of $\mathbb{R}$, there exists some N such that for all $n \in \mathbb{N}$, $1/n < \varepsilon$ meaning $1/n \in V_{\alpha^*}$. Asides from the points $1/n$ where $n \leq N$, all points are elements of V_{α^*} and points not in V_{α^*} are contained in at least one other open set V_α. Taking the union of V_{α^*} and a finite number of sets containing the rest of the points, we've constructed a finite open cover. So, K is compact by this construction.

Problem 7

(2.30) Imitate the proof of Theorem 2.43 to obtain the following result: If $\mathbb{R}^k = \bigcup\limits_{n=1}^{\infty} F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.

Solution

For each $n \in \mathbb{N}$ construct $G_n = \mathbb{R}^k \setminus F_n$. We know that G_n is open for all $n \in \mathbb{N}$ and further that

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \mathbb{R}^k \setminus F_n = \emptyset.$$

This means that each G_n isn't dense in $\mathbb{R}^k$. Hence there exist some $N \in \mathbb{N}$ such that $\bar{G}_N \neq \mathbb{R}^k$. However, $F_N^\circ = \mathbb{R}^k \setminus \bar{G}_N$, and so $F_N^\circ \neq \emptyset$, i.e., there's some F_N with nonempty interior F_N°.

Problem 8

(3.8) If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Solution

Without loss of generality, assume $\{b_n\}$ is increasing[1]. Because $\{b_n\}$ is bounded it also converges. Let $\lim\limits_{n\to\infty} b_n = b$. We know that the series $\sum a_n (b - b_n)$ converges since $\lim\limits_{n\to\infty} a_n (b - b_n) = 0$ and $\{b_n\}$ is an increasing sequence combined with the fact that $\sum a_n$ converges. Because the sum/difference of two convergent series is again convergent and

$$\sum a_n (b - b_n) = b \sum a_n - \sum a_n b_n,$$

$\sum a_n b_n$ converges.

[1] We can replace $\{b_n\}$ with $\{-b_n\}$ in the case that $\{b_n\}$ is decreasing.

Problem 9

(3.25) Let X be the metric space whose points are the rational numbers, with the metric $d(x,y) = |x-y|$. What is the completion of this space?

Solution

We know that the metric is the restriction of the usual Euclidean metric on $\mathbb{R}$ (this extension of d makes $\mathbb{R}$ complete).

Problem 10

(4.1) Suppose f is a real function defined on $\mathbb{R}$ which satisfies

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}$. Does this imply that f is continuous.

Solution

No! Let $f(x) = 0$ for all $x \neq 0$ and $f(0) = c$ where $c \in \mathbb{R}$. Clearly the above condition holds, but f isn't continuous!

Problem 11

(4.8) Let f be a uniformly continuous on the bounded set E in $\mathbb{R}$. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Solution

Let's assume that f isn't bounded on E. Then there exists a sequence $\{x_n\}$ where each $x_n \in E$ for $n \in \mathbb{N}$ such that $|f(x_n)| > n$. Because $\{x_n\}$ is a sequence in a bounded set, the sequence itself is bounded and therefore has a convergent subsequence. Call this subsequence $\{x_{\alpha_n}\}$. Because this subsequence converges, it's a Cauchy sequence in E and therefore $\{f(x_{\alpha_n})\}$ must also be a Cauchy sequence in E because f is uniformly continuous on E. However, $|f(x_{\alpha_n})| > \alpha_n > n$ meaning f can't be a Cauchy sequence, thus we have a contradiction and f must be bounded on E.

To show that we need the boundedness of E to arrive at the conclusion, consider the identity function $\mathrm{id} : \mathbb{R} \to \mathbb{R}$. Clearly id is uniformly continuous but $\mathbb{R}$ isn't bounded, yet id isn't unbounded, thus we need E to be a bounded set in $\mathbb{R}$.

Problem 12

(4.12) A uniformly continuous function of a uniformly continuous function is uniformly continuous. State this more precisely and prove it.

Solution

Proposition 2. *Let $f : X \to Y$ and $g : Y \to Z$ be uniformly continuous functions on metric spaces X, Y, and Z. The composition $gf : X \to Z$ is also uniformly continuous.*

Problem 12 continued on next page. . .

Proof. By the uniform continuity of f, for all $\varepsilon' > 0$, there exists a delta $\delta > 0$ such that

$$d_Y\left(f(x_1), f(x_2)\right) < \varepsilon' \text{ whenever } d_X\left(x_1, x_2\right) < \delta$$

for all $x_1, x_2 \in X$. Similarly, because g is uniformly continuous, for all $\varepsilon > 0$, there exists a $\varepsilon' > 0$ such that

$$d_Z\left(g(f(x_1)), g(f(x_2))\right) < \varepsilon \text{ whenever } d_Y\left(f(x_1), f(x_2)\right) < \varepsilon'$$

which implies this is the case whenever $d_X\left(x_1, x_2\right) < \delta$ for any $x_1, x_2 \in X$. Thus, gf is uniformly continuous. □

Problem 13

(4.14) Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Solution

Consider the function $g(x) = x - f(x)$. Because g is a composition of f and the addition map, g is continuous. Let's assume that $g(x) \neq 0$ for any $x \in I$. At the point $x = 0$, $g(0) = -f(0) < 0$ by our assumption and at the point $x = 1$, $g(1) = 1 - f(1) > 0$. However, by the intermediate value theorem, there exists a $x \in (0, 1)$ such that $g(x) = 0$ which implies $x - f(x) = 0$ which leads to $f(x) = x$ for some $x \in I$.

Problem 14

(5.4) If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots C_n x^n = 0$$

has at least one real root between 0 and 1.

Solution

Let $P(x) = C_0 X + \frac{C_2}{2} x^2 + \cdots + \frac{C_n}{n+1} x^{n+1}$. At $x = 0$ and $x = 1$, $P(x) = 0$. Hence, by the mean value theorem, there exists some $x \in (0, 1)$ such that

$$\frac{dP}{dx} = C_0 + \cdots + C_n x^n = 0.$$

Problem 15

(5.7) Suppose $f'(x)$, $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

Solution

By the alternative definition of the derivative, the limits

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} \quad \text{and} \quad \lim_{t \to x} \frac{g(t) - g(x)}{t - x}$$

exist. Rewriting the left hand side of the equation in the problem statement,

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{f(t) - f(x)}{f(t) - f(x)} = \lim_{t \to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \frac{\lim\limits_{t \to x} \frac{f(t)-f(x)}{t-x}}{\lim\limits_{t \to x} \frac{g(t)-g(x)}{t-x}} = \frac{f'(x)}{g'(x)}.$$

Problem 16

(5.20) Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector-valued functions.

Solution

Theorem 3. *Let $f : [a, b] \to \mathbb{R}^n$ be a continuous map, k be natural number, $f^{(k-1)}$ be continuous on $[a, b]$, and $f^{(n)}(t)$ exist for all $t \in (a, b)$. Define the polynomial*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k-1)}(\alpha)}{k!}(t - \alpha)^k.$$

Let $\alpha \neq \beta$ where both are in the interval $[a, b]$. There exists an $\alpha < x < \beta$ such that

$$\|f(\beta) - P(\beta)\| \leq \frac{\left\|f^{(n)}(x)\right\|}{n!}|\beta - \alpha|^n.$$

Proof. Let $\zeta = f(\beta) - P(\beta)$ and $\phi(t) = \zeta \cdot f(t)$. Clearly ϕ satisfies the assumptions in Theorem 3. By Taylor's theorem,

Problem 16 continued on next page...

$$
\begin{aligned}
\phi(\beta) &= \sum_{k=0}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}(\beta-\alpha)^k + \frac{\zeta \cdot f^{(n)}(t)}{n!}(\beta-\alpha)^n \\
&= \zeta \cdot P(\beta) + \frac{\zeta \cdot f^{(n)}(t)}{n!}(\beta-\alpha)^n \\
\phi(\beta) - \zeta \cdot P(\beta) &= \frac{\zeta \cdot f^{(n)}(t)}{n!}(\beta-\alpha)^n \\
\zeta \cdot (f(\beta) - P(\beta)) &= \\
\zeta \cdot \zeta &= \\
\|\zeta\|^2 &= \frac{\zeta \cdot f^{(n)}(t)}{n!}(\beta-\alpha)^n \\
&= \left| \frac{\zeta \cdot f^{(n)}(t)}{n!}(\beta-\alpha)^n \right| \\
&\leq \frac{\|\zeta\| \left\| f^{(n)}(t) \right\|}{n!}|\beta-\alpha|^n.
\end{aligned}
$$

By definition of ζ,

$$
\|f(\beta) - P(\beta)\| \leq \frac{\left\| f^{(n)}(t) \right\|}{n!}|\beta-\alpha|^n.
$$

□