A Very Circular Integral

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How do we calculate the following integral?

$$\int_{0}^{x} \sqrt{1-t^2}dt$$

One thing about the integral $\sqrt{1-t^2}$ is that it corresponds to the implicit equation $x^2 + y^2 = 1$ where we solve for y and replace x with t. Then, the integral $\int_0^x \sqrt{1-t^2}dt$ is the area described by Figure 1.

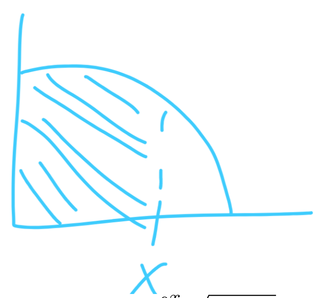

Figure 1: The integral $\int_0^x \sqrt{1-t^2}dt$ visualized.

We then can divide this area into two parts; an slice of the circle and a triangle as depicted in Figure 2.

Figure 2: The integral $\int_0^x \sqrt{1-t^2}dt$ split into a slice of the unit circle and a triangle.

Clearly then, the area of the triangle is $\frac{1}{2}\left(x\sqrt{1-x^2}\right)$. If we let θ be the angle between the y-axis and the hypotenuse of the constructed triangle, the area of the slice would be $\frac{1}{2}\theta$. However, notice that we can construct the similar triangle as depicted in Figure 3.

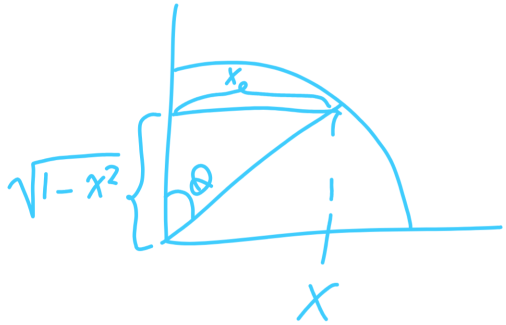

Figure 3: The construction of the similar triangle.

Using elementary trigonometry, $\tan(\theta) = \frac{x}{\sqrt{1-x^2}}$, and so $\theta = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$. Therefore

$$\int_0^x \sqrt{1-t^2}dt = \frac{x\sqrt{1-x^2} + \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)}{2} + C. \tag{1}$$